King Fahd University of Petroleum & Minerals
Department of Information and Computer Science
Dr. Husni Al-Muhtaseb

Question	1 - 10	11	12	13	14	15	Total
Max	50	30	4	6	10	10	110
CLO*	2, 3	2	3	3	2	2	
Earned							

* Course Learning outcomes

Exam 2 Sample Solution of Version 01

Questions 1-10: [5 Points Each] Choose the correct answer from the options listed. Choose only one answer. **Mark the correct answer by placing an "X" in its box.**

Answer Table for questions 1-10. Mark the correct answer by placing an "X" in its box.

	a.	b.	c.	d.
1.				X
2.		X		
3.		X		
4.			X	
5.	X			
6.	X			
7.			X	
8.			X	
9.				X
10.				X

Question 1: [5 Points] Sequences and Summations [CLO 2]

Which of the following sequences is described, as far as it goes, by an explicit formula ($n \geq 0$) of the form $g_n = \left\lfloor \frac{n}{k} \right\rfloor$, where $\lfloor \quad \rfloor$ is the floor function and k is a positive integer.

(a) 0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2

(b) 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3

(c) 0, 0, 0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4

(d) 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3

Question 2: [5 Points] Sequences and Summations [CLO 2]

Which of the following sum is gotten from $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$ by the change of variable $j = i + 1$?

(a) $\sum_{j=2}^{n} \frac{j-1}{(n-j-1)^2}$

(b) $\sum_{j=2}^{n} \frac{j-1}{(n-j+1)^2}$

(c) $\sum_{j=2}^{n} \frac{j}{(n-j+1)^2}$

(d) $\sum_{j=2}^{n} \frac{j}{(n-j-1)^2}$

Question 3: [5 Points] Sequences and Summations [CLO 2]

The first term of an arithmetic progression is $a_0 = 1$ and the term a_{11} is $a_{11} = 23$. Find the term a_{20} of the progression.

(a) $a_{20} = 39$ (b) $a_{20} = 41$

(c) $a_{20} = 42$ (d) There is not enough information.

Question 4: [5 Points] Induction and Recursion [CLO 3]

Which of the following statements is false?

(a) If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height $h(T) = 1 + \max(h(T_1), h(T_2))$.

(b) If T is a full binary tree T, then The number of vertices $n(T) \le 2^{h(T)+1} - 1$, where $h(T)$is the height of the tree T.

(c) A full binary tree cannot be a single vertex.

(d) A full binary tree is a tree in which every node other than the leaves has two children.

Question 5: [5 Points] Induction and Recursion [CLO 3]

We are going to prove by induction that for all integers $k \ge 1$, $\sqrt{k} \le \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}$. Clearly this is true for $k = 1$. Assume the Induction Hypothesis (IH) that

$$\sqrt{n} \le \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

Which is a correct way of concluding this proof by induction?

(a) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \ge \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$

(b) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + \frac{1}{\sqrt{n+1}} = \sqrt{n+1} + 1 \ge \sqrt{n+1}$

(c) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \ge \sqrt{n+1} + \frac{1}{\sqrt{n+1}} \ge \sqrt{n+1}$

(d) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + \frac{1}{\sqrt{n}} = \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n}} \ge \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$

Question 6: [5 Points] The Pigeonhole Principle [CLO 3]

There are K people in a room; each person picks a day of the year to get a free dinner at a fancy restaurant. K is such that there must be at least one group of six people who select the same day. What is the smallest such K if the year is a leap year (366 days)?

(a) 1831 (b) 1833 (c) 1830 (d) 1832

Question 7: [5 Points] Induction and Recursion [CLO 3]

We Suppose $b_1, b_2, b_3, \cdots$ is a sequence defined by $b_1 = 3$, $b_2 = 6$, $b_k = b_{k-2} + b_{k-1}$ for $k \geq 3$. Prove that b_n is divisible by 3 for all integers $n \geq 1$. Regarding the induction hypothesis, which is true?

(a) Assuming this statement is true for $k \leq n$ is enough to show that it is true for $n + 1$ and no weaker assumption will do since this proof is an example of "strong induction."

(b) Assuming this statement is true for n, n−1, and n−3 is enough to show that it is true for $n + 1$ and no weaker assumption will do since you need three consecutive integers to insure divisibility by 3.

(c) Assuming this statement is true for n and $n - 1$ is enough to show that it is true for $n + 1$.

(d) Assuming this statement is true for n is enough to show that it is true for $n + 1$.

Question 8: [5 Points] Counting and the Pigeonhole Principle [CLO 3]

Let S = {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21}. What is the smallest integer N > 0 such that for any set of N integers, chosen from S, there must be two distinct integers that divide each other?

(a) 10 (b) 7 (c) 8 (d) 9

Question 9: [5 Points] Permutations and Combinations [CLO 3]

The owner of a pizzeria prepares every pizza by always combining 4 different ingredients. How many ingredients does he need, at least, if he would like to offer 30 different pizzas in the menu?

(a) 8 (b) 6 (c) 9 (d) 7

Question 10: [5 Points] Permutations and Combinations [CLO 3]

Which of the following statements is false:

(a) There are $(10^3)(5)=5000$ strings of four decimal digits end with an even digit.

(b) There are (10)(9)(8)(7)=5040 strings of four decimal digits do not contain the same digit twice.

(c) There are (9)(4)=36 strings of four decimal digits have exactly three digits that are 9s.

(d) All the other answers are false.

Question 11: [30 Points] Sequences, and summations [CLO 2]

Indicate whether the given sentence is true or false. In the answer column write either ✓ for "true" or ✗ for "false".

Statement	Answer
1. A string is a finite sequence of characters from a finite set (an alphabet).	✓
2. An *arithmetic progression* is a sequence of the form: $a, ar, ar^2, \dots, ar^n, \dots$ where the initial term a and the common ratio r are real numbers.	✗
3. A *geometric progression* is a sequence of the form: $a, a+d, a+2d, \dots, a+nd, \dots$ where the initial term a and the common difference d are real numbers.	✗
4. If x is real and n is integer then $\lfloor x+n \rfloor \neq \lfloor x \rfloor + n$. (Where $\lfloor \quad \rfloor$ is the floor function).	✗
5. The subset of all real numbers that fall between 0 and 1 is countable.	✗
6. The formula for the sequence 1, 1/2, 1/4, 1/8, 1/16, … is $a_n = 1/4^n$, $n = 0, 1, 2, \dots$	✗
7. We can pick 11 players from 21 candidates in $\binom{21}{11}$ ways.	✓
8. $\sum_{k=1}^{n}(ak+d) = (a\sum_{k=1}^{n} k) + (nd)$.	✓
9. $\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4}(k+1)^2$	✓
10. There are 1000 positive integers devisable by 9 between 1000 and 9999 inclusive.	✓
11. The set of positive rational numbers is countable.	✓
12. If A and B are countable sets, then $A \cup B$ is also countable.	✓
13. Mathematical induction can be used to prove mathematical statements that assert a property is true for all positive integers such as “for every positive integer n: $n! \leq n^n$.”	✓
14. The nth term of the sequence 5, 6, 10, 28, 124, 724, 5044, … is $n! + 4$.	✓
15. There are 8! permutations of the letters ABCDEFGH contain the string “ABC”	✗

Question 12: [4 Points] Binomial Coefficients and Identities [CLO 3]

The row of Pascal’s triangle containing the binomial coefficients $\binom{10}{k}$, $0 \leq k \leq 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal’s identity to produce the row immediately following this row in Pascal’s triangle.

Solution:

By Pascal's identity, the first half of next row is

1 1+10 10+45 45+120 120+210 210+252 · · ·

With the rest determined by symmetry. Thus, the next row is

1 11 55 165 330 462 462 330 165 55 11 1

Question 13: [6 Points] Binomial Coefficients and Identities [CLO 3]

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$? You may need to use the binomial formula $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k$. Show your answer using combination and multiplication of numbers (no need to calculate a final answer).

Solution: Using the binomial formula taking a=2x, b=-3y, n=200 and k=99 we have that $x^{101}y^{99}$ term is $\binom{200}{99}(2x)^{101}(-3y)^{99} = -\binom{200}{99}2^{101}3^{99}x^{101}y^{99}$

Hence the desired coefficient is $-\binom{200}{99}2^{101}3^{99}$

Question 14: [10 Points] Induction [CLO 2]

Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.

Solution:
We use mathematical induction.
Basis step: for $n = 1$, the equation states that $1 \cdot 1! = (1 + 1)! - 1$, and this is true because both sides of the equation evaluate to 1.
Inductive hypothesis: Assume that $p(k) = 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$ for some positive integer k. We need to prove that $p(k+1)$ is true. That is
$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1)!(k + 2) = (k + 2)! - 1$
Inductive step:
$p(k) = 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$
We add $(k + 1)(k + 1)!$ to $p(k)$ to find that
$1 \cdot 1! + 2 \cdot 2! + \cdots + (k + 1) \cdot (k + 1)! = (k + 1)! - 1 + (k + 1)(k + 1)!$ The right hand side equals
$(k + 1)!(k + 2) - 1 = (k + 2)! - 1$. This establishes the desired equation also for $k + 1$, and we are done by the principle of mathematical induction

Question 15: [10 Points] Induction [CLO 2]
Use strong induction to show that every positive integer *n* can be written as a sum of distinct powers of two, that is, as a sum of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and so on.

Hint: for the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.

Solution:
Let $P(n)$ be the claim that *n* can be written as a sum of distinct powers of two. We show that $P(n)$ is true for all positive integers *n*.

Basis step: As $1 = 2^0$, we have that $P(1)$ is true.

Inductive hypothesis: Assume for a positive integer *k* that $P(i)$ is true for all $1 \le i \le k$.

Inductive step: We consider two cases, namely when $k + 1$ is even and when $k + 1$ is odd. If $k + 1$ is even, then $(k + 1)/2$ is an integer, and by the inductive hypothesis, we can express $(k + 1)/2$ by a sum of distinct powers of two. We can then multiply this sum by 2, which simply increases the exponent of each power of two by 1, so this is again a sum of distinct powers of two that is equal to $k + 1$.

When $k + 1$ is odd, we have that *k* is even. By the inductive hypothesis, we can express *k* as a sum of distinct powers of two. However, since *k* is even, the sum cannot contain $2^0 = 1$. Thus, we can add 2^0 to this sum, which remains a sum of distinct powers of two, and equals $k + 1$.

Thus, in both cases, we can express $k + 1$ as a sum of distinct powers of two, so when $P(i)$ is true for all $1 \le i \le k$, we have that $P(k + 1)$ is true. Since $P(1)$ is true, this means that the claim is true for all positive integers, as show by strong induction.